# Remarks on transition in a round tube

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This article has a twofold purpose: (1) to analyse the available theoretical and experimental knowledge concerning flow in the inlet region of a smooth round tube, and (2) to point out that the $e^9$ amplification factor method apparently predicts natural transition correctly over a significant fraction of the entire inlet length of the tube. The successful prediction indicates, but does not prove, that flow in a smooth round tube becomes turbulent at higher Reynolds numbers because transition occurs in the inlet length—not in the fully developed Poiseuille régime. The close agreement between theory and a test result obtained by Pfenninger indicates that the $e^9$ method is valid for a wide variety of flows having $x$ Reynolds numbers of transition ranging from 570,000 to 40 million. The results are applicable to both plane and axially symmetric flows.

## 1. Introduction

In 1956 Smith & Gamberoni showed by analysis of a large number of experiments that boundary-layer transition will occur when Tollmien–Schlichting waves reach an apparent amplification ratio of about $e^9$. One experiment, not treated in the original work, was that performed by Pfenninger (1950, 1951 *a*, *b*) at the Northrop Aircraft Co., in which he carefully measured the transition point in a long (59 ft.) tube of 2 in. diameter containing a flow with very low turbulence. This type of test is of particular interest because the problem of transition in a round tube has been paradoxical for many years. Theory predicts that fully developed parabolic flow is stable with respect to small rotationally symmetric disturbances at all Reynolds numbers; yet the flow is known to become turbulent at rather low values of the Reynolds number. The reasons Pfenninger's tube-test was not included in the original study were two: (1) the flow was not of a boundary-layer type except very near the entrance, and (2) no amplification rate chart applicable to this kind of flow was in existence. Subsequently, efforts have been made to analyse the flow, and several observations concerning the state of knowledge about this problem were made that are believed to be of general interest. This paper then has a twofold purpose: to report that the $e^9$ factor does seem to hold for the tube flow studied and, secondly, to call attention to and discuss the unsatisfactory state of knowledge concerning this classical yet still paradoxical flow problem.

## 2. Experimental data

Many tests have been made to learn the tube Reynolds number $R_d = \bar{U}d/\nu$ ($\bar{U}$ = mean velocity, $d$ = diameter, $\nu$ = kinematic viscosity) at which the flow becomes turbulent; for example, see Goldstein (1938), Schlichting (1955), or Prandtl & Tietjens (1934). Yet very little study has been made of the nature of the flow in the initial portion of the tube, the inlet length, and one wishing to study the problem in detail will find that a complete set of reliable test data does not exist. Three of the principal sets of data will be mentioned here. One is due to Nikuradse (see Prandtl & Tietjens 1934). It is complete in a static sense, consisting of a full set of velocity profiles covering the entire inlet length. However, all the results are presented in two small graphs with neither explanation nor description of the test apparatus. Therefore the data have a low degree of accuracy, if for no other reason than the small size of the figures.

Another set is Pfenninger's data. In this case great care was used in taking measurements, the apparatus was very good, and the scale was large. Nevertheless, the data are not entirely satisfactory, because the tests were in the nature of spot checks rather than a complete, systematic survey. For example, transition location is reported only for one tube Reynolds number. No curve of $R_x$ (the Reynolds number based on $x$) for transition versus $R_d$ is given. Several velocity profiles were measured, but these were taken under different conditions and in slightly different apparatus from those for which the transition measurement was made. In addition, the Northrop data cover only a small fraction of the entire inlet length.

Recently a third set of data have become available: the results of tests run by Reshotko (1958) in an extremely fine experimental arrangement located underground in a nearly isothermal environment. But again the data cannot be called complete. The purpose of the tests was to study the stability of fully developed Poiseuille flow. Consequently, only the downstream portions of the inlet region were measured, and the Reynolds number could not be increased sufficiently to induce transition in the inlet length. (Another experimental investigation has been performed by Leite (1959), its purpose and interest being quite similar to Reshotko's. However, his data will not be cited here, because Reshotko's apparatus was considerably superior, and because Leite covered only the fully developed flow régime. Nevertheless, his results supply a great deal of information on the stability of fully developed flow.)

Figure 1 compares transverse velocity profiles according to Nikuradse, Pfenninger and Reshotko. In order to prepare this figure, it was first necessary to cross-plot Pfenninger's and Reshotko's results after the fashion of figure 13 of Prandtl & Tietjens (1934). The profiles so found are seen to have none too good agreement, differing by as much as 9 % at the centre. Notice also that Pfenninger's data cover only the first part of the inlet length, whereas Reshotko's cover only the last part; and since there is no overlap, they cannot be compared directly. The entire length corresponds to a value of $x/aR_a$ of about 0·26. (Because $x/aR_a$ has been the most common length parameter in the past, it is used here: $R_a$ is the tube Reynolds number $\overline{U}a/\nu$, where $a$ is the radius—not the diameter—and $\overline{U}$ is the mean velocity.) The question of whether the differences are attributable to graph-reading accuracy, experimental accuracy, differences in entrance conditions, or other factors cannot be answered. Some of the same data are shown in a different and perhaps clearer fashion in figure 2. In figure 3 the core velocities

$U_c$ are compared. Again, Nikuradse's and Pfenninger's data are seen to be in only fair agreement. Reshotko's data are mostly in good agreement with Nikuradse's, but they too fail to overlap Pfenninger's. This figure shows clearly that Pfenninger's tests covered only a small fraction of the entire inlet length.

In summary, only three sets of detailed measurements of mean velocities exist. One, Nikuradse's, is complete but of uncertain accuracy; the other two are of good accuracy but incomplete for the present interest.

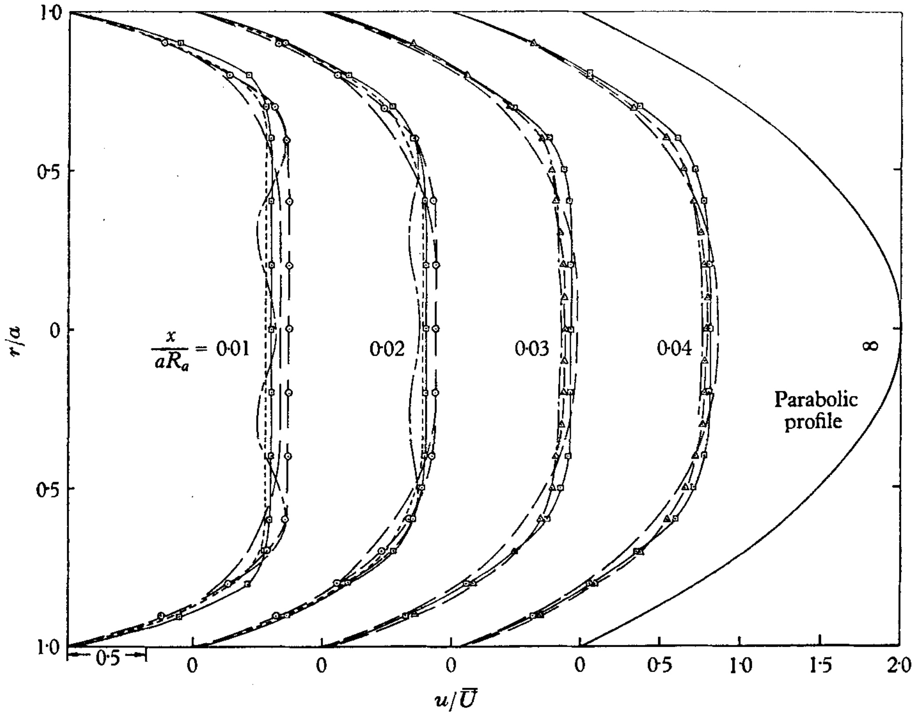


FIGURE 1. Measured and calculated laminar velocity profiles at several positions in the inlet portion of a round tube. Experimental: —□—□—□, Nikuradse; —○—○—○, Pfenninger (1951*a*, Fig. 3); —△—△—△, Reshotko. Theoretical: — — — —, Langhaar; — - —, Punnis; - - - - -, Tatsumi.

## 3. Recent theoretical studies

Between theory (Lin 1955; Corcos & Sellars 1959) and experiment (Leite 1959), it has become fairly well established that the asymptotic (parabolic) flow in a round tube is stable to rotationally symmetric *small* disturbances at all Reynolds numbers. Consequently, if stability theory is to explain natural transition for flow of low turbulence in a smooth round tube, it must explain it either by consideration of the inlet region alone or else by consideration of disturbances that lack rotational symmetry, in which case the Poiseuille régime may also enter the picture. Since it is known that transition can occur in the inlet length, recent efforts at explanation have dealt with flow in this portion.

In order to apply stability theory, accurate primary velocity profiles must be available. The other results of the older theoretical attempts (Goldstein 1938) to

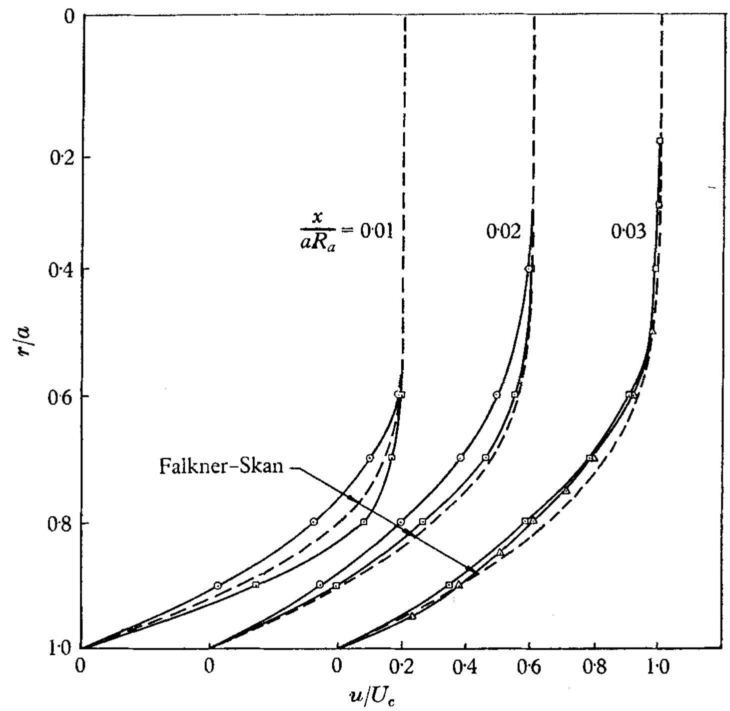


FIGURE 2. Comparison of measured velocity profiles with calculations by Falkner–Skan piecewise method. —○—○—○, Pfenninger (1951*a*, Fig. 3); —□—□—□, Nikuradse; —△—△—△, Reshotko.

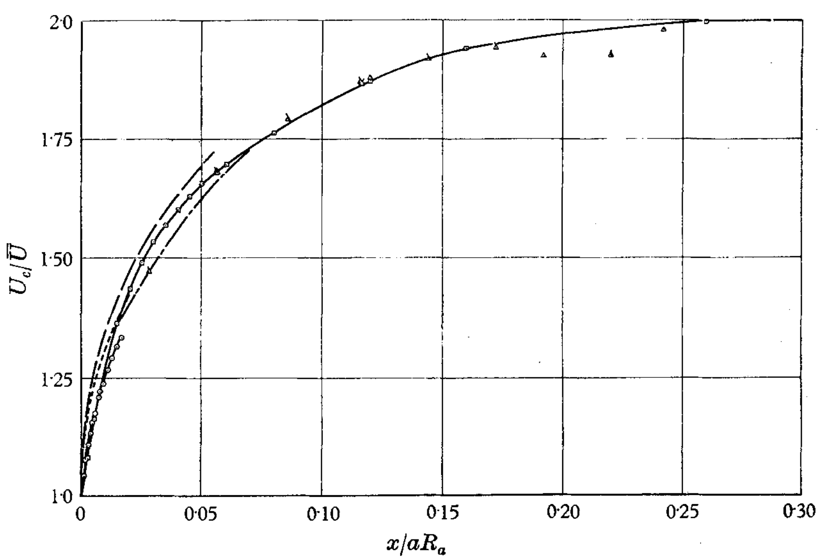


FIGURE 3. Comparison of several measured and calculated values of velocity in the core of a round tube. —□—, Nikuradse; —○—, Pfenninger (1951*b*, Fig. 5, $U_{1,0} = 26{\cdot}28$ m/s). △, $R_a = 2050$, △ $R_a = 3800$, △ $R_a = 18{,}000$, Reshotko. - - - - -, Tatsumi; —·—, Punnis; — — —, Langhaar.

find them are unsuitable, and we shall consider only the three most recent attempts to solve the problem. Those to be considered are by Langhaar (1942), Tatsumi (1952), and Punnis (1947). Results of these attempts are presented in figures 1 and 3. It is seen that the theoretical values are not only in disagreement among themselves but also in disagreement with the two sets of experimental results. For many purposes the agreement indicated might be considered good, but for purposes of stability calculations the accuracy is inadequate.

The only calculations of the stability of these inlet profiles are those by Tatsumi, who computed the neutral stability loops for several values of $x/aR_a$. His computed values of $R_{\delta^*_{\text{crit}}}$ $(=U\delta^*/\nu$, where $\delta^*$ is the displacement thickness and $U$ is the edge velocity) are shown in figure 4. In Pfenninger's test the measured location of transition was at $x/aR_a = 0{\cdot}017$. Hence Tatsumi's stability calculations did not cover a sufficient portion of the inlet region to make possible calculations of wave amplification, even if he had computed complete stability maps—which he did not.

## 4. A rough calculation of the neutral-stability and transition loci

By now it should be obvious why Pfenninger's tube-test was not included in the original correlation studies. Since both experimental and theoretical data needed for amplification studies of inlet flow were found wanting, a decision was made to apply ordinary methods of boundary-layer calculation to learn whether or not they might produce a useful result. The method chosen was exactly that used in the earlier studies, the Falkner–Skan piecewise method (Smith 1956*b*), using Mangler's transformation. The boundary-layer profiles resulting from the calculation are characterized by Hartree's $\beta$. Once the variation of $\beta$ along the axis of the tube is established, the stability or degree of instability of the flow can be established from Pretsch's set of amplification rate curves (see Smith & Gamberoni 1956).

The tube was treated as one of decreasing diameter, whose effective area decreased inversely as the core velocity supplied by the experiment. Figure 2 shows the boundary-layer profiles calculated by the piecewise method. Both this figure and figure 1 show that the profiles have about the same level of accuracy at $x/aR_a = 0{\cdot}03$ as those computed by the more involved methods of calculation.

Tatsumi calculated stability of the flow only for values of $x/aR_a \leqslant 0{\cdot}004$. In this range the piecewise method has very high accuracy and is clearly much more rapid than Tatsumi's method.

It was not known whether the location of the neutral-stability curve was very sensitive to the shape of the core-velocity distribution. To learn the answer, neutral stability for *two-dimensional* disturbances was calculated using two different core-velocity distributions, Pfenninger's and Nikuradse's. The results are shown in figure 4. The two different distributions cause appreciable difference in the calculated stability of the flow.

How does the stability calculated indirectly from the core-velocity-distribution data agree with that calculated directly from experimental velocity profiles? Pfenninger measured carefully the velocity profiles at a station 39·64 ft. from the entrance. Because he tested at several velocities, the profiles cover a small range of values of $x/aR_a$. The neutral stability of four of these profiles covering the $x/aR_a$ range were calculated by means of Lin's short formula. The necessary velocity and derivatives were obtained by fitting a 20-point least-squares cubic to the boundary layer in the range $0 \leqslant y/\delta^* \leqslant 1$. The results, plotted in figure 4, show that the direct calculation of stability from the experimentally measured boundary-layer profiles produces considerably higher values of $R_{\delta^*_{\text{crit}}}$ than those determined indirectly from the core-velocity distribution. Furthermore, beyond $x/aR_a$ of about 0·015 the stability of the experimental velocity profiles undergoes a rapid increase, while that determined from the core-velocity distribution begins to decrease.

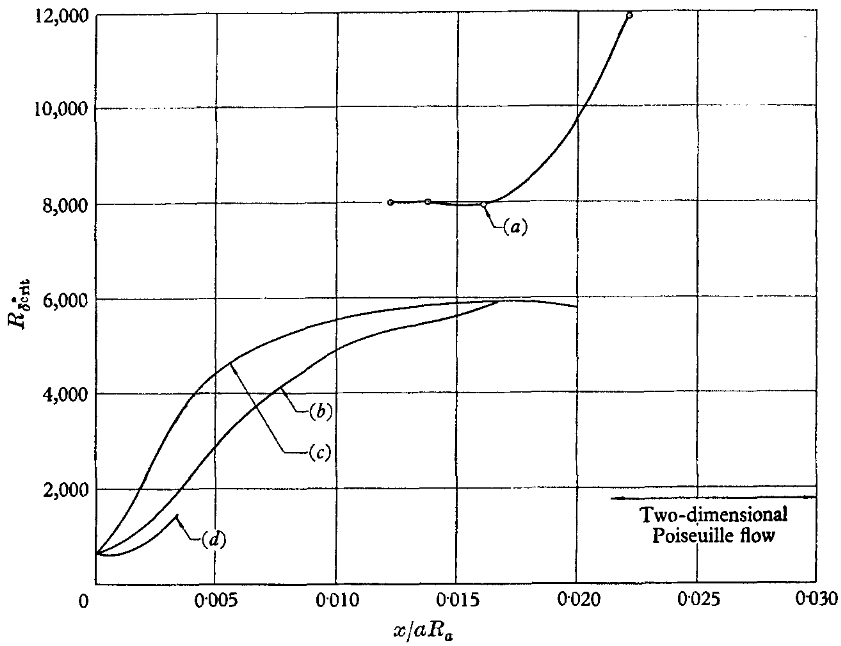


FIGURE 4. Variation of neutral stability along the inlet length of a round tube according to several calculations. (*a*) Experimental, direct computation from measured velocity profiles (Pfenninger 1951*a*, Fig. 3) using Lin's short formula. The flow is treated as if it were two-dimensional. (*b*) Pfenninger data (1951*b*, Fig. 5, $U_{1,0} = 26{\cdot}28$ m/s) calculated by piecewise method (Smith 1956*b*) and Pretsch charts (Smith & Gamberoni 1956). (*c*) Nikuradse data, calculated by piecewise method and Pretsch charts. (*d*) Tatsumi.

Reshotko's measurements are in general agreement with these other data but his velocity traverses included too few points for the difficult problem of computing stability of an experimentally measured velocity profile.

Tatsumi's calculation is also shown in figure 4. It represents the stability of rotationally symmetric disturbances, however, whereas the other data represent the stability of two-dimensional disturbances. For its interest, the critical Reynolds number of two-dimensional Poiseuille flow is also included. This value has been obtained from the commonly accepted value of $R = 5300$, where $R$ is based on the half-width of the channel and the velocity at the centre of the channel. With respect to the other data this value is surprisingly low.

Figure 5 shows the point of transition calculated by the $e^9$ method, the neutral-stability locus according to Tatsumi, and the locus according to the present method of calculation. For purposes of calculating transition to the very end of the inlet length, Nikuradse's core-velocity distribution was used. Also to be seen on the figure is the experimentally measured transition point supplied by Pfenninger. The calculated point nearby is computed in the same way as the full curve based on Nikuradse's data, except that it uses the Pfenninger core-velocity distribution existing during the test when the transition point was observed. The

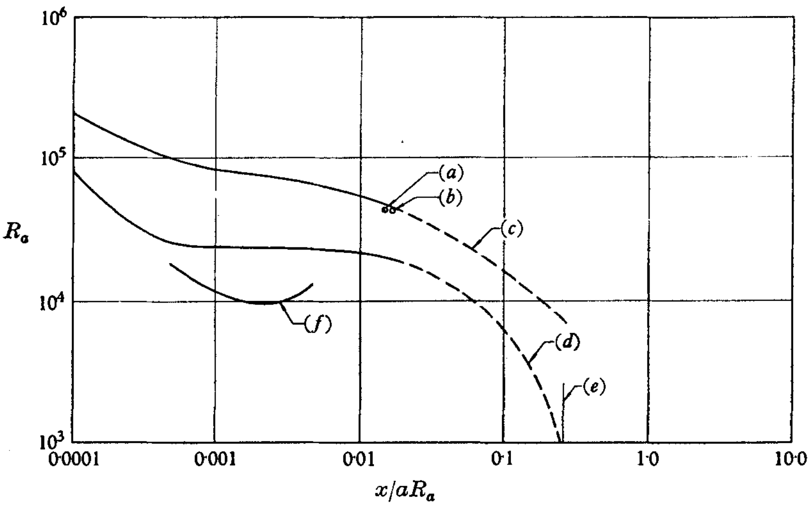


FIGURE 5. Transition and neutral-stability loci according to several calculations. Also shown is Pfenninger's transition measurement. (*a*) Calculated, using data from Pfenninger (1951*b*) $U_{1,0} = 26{\cdot}28$ m/s. (*b*) Experimental Pfenninger (1951*b*) $U_{1,0} = 26{\cdot}28$ ms. (*c*) $e^9$ transition curve, Nikuradse core velocity. (*d*) Neutral curve, Nikuradse core velocity. (*e*) End of inlet length. (*f*) Neutral curve, Tatsumi.

most interesting feature of this figure is the close agreement of the predicted and the experimental transition point. Moreover, the small difference between the location of the circled Pfenninger point and the Nikuradse locus indicates that the differences in core-velocity distribution do not cause significant changes in the calculated location of transition, even though the differences in neutral-stability locus were appreciable (figure 4). Since theory and Pfenninger's test are in good agreement, and since boundary-layer calculations gain in accuracy near the entrance, it is probable that the curve of $R_a$ for transition versus $x/aR_a$ is fairly accurate for values of $x/aR_a$ less than about 0·02. At higher values of $x/aR_a$, the predicted transition curve should rapidly develop major errors. Clearly, this portion of the curve is very conservative, as can be realized by the following consideration. Farther forward in the inlet, the flow is accelerating, and as a result it is a flow having positive values of Hartree's $\beta$ (or Pohlhausen's $\Lambda$). Downstream of the inlet length, where the flow has become parabolic, the velocity gradient becomes zero. Obviously, boundary-layer calculations would predict an ordinary Blasius boundary layer of low stability for this region. But from the correct calculations we know that the flow in this region has infinite stability, at least for axially symmetric disturbances. In spite of the ultra-conservative nature of the calculations for the larger values of $x/aR_a$, the curve shows that at $R_a = 7{\cdot}1 \times 10^3$ transition will occur at the very end of the inlet length.

## 5. Miscellaneous remarks on the transition phenomenon

If all disturbances are damped in the parabolic region† and if the flow succeeds in reaching this region in the laminar state, it may remain laminar forever. Conversely, if natural transition does occur in the tube, it will occur in the inlet portion. In view of the very conservative nature of the calculations for

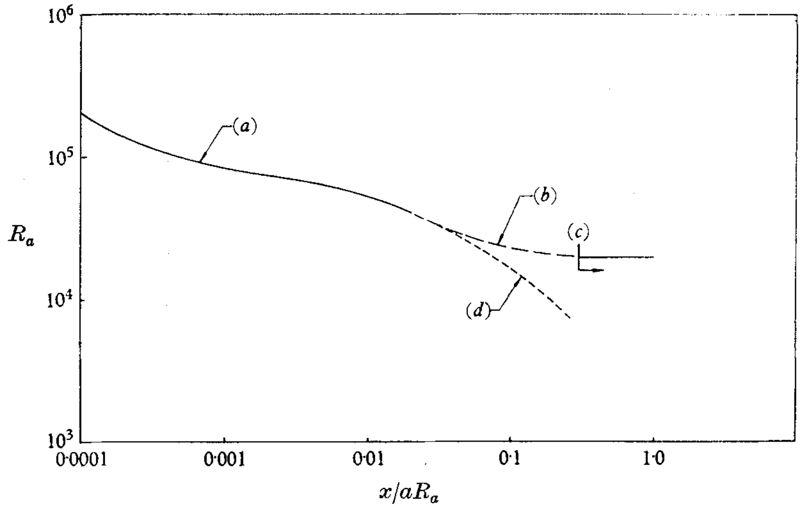


FIGURE 6. General nature of the transition locus in the $x/aR_a$ *vs* $R_a$ plane. (*a*) Transition locus (this part can be calculated with acceptable accuracy). (*b*) More nearly correct result. (*c*) Parabolic flow. (*d*) Conservative boundary-layer calculation.

$x/aR_a > 0{\cdot}02$, it is clear from the figure that laminar flow can exist to the end of the inlet length, at least to $R_a = 7{\cdot}1 \times 10^3$, that is, to $R_d = 14{\cdot}2 \times 10^3$. Experiment has shown that the flow can remain fully laminar to values of $R_d$ of about 30,000.

The more nearly correct relations between $R_a$ and $x/aR_a$ for transition should be qualitatively as sketched in figure 6. Beyond the inlet, $R_a$ for transition becomes constant, and $R_x$ for transition becomes infinite.

In the $R_a$ *vs* $\bar{U}x/\nu$ plane the transition locus should look somewhat like the curve in figure 7. Although this figure is principally presented for schematic purposes, it is actually calculated from the upper line of figure 6, and in fact the curve to the right of $R_a = 40{,}000$ represents the same values as those shown in figure 5, that is, the portion to the right of Pfenninger's point is the theoretical $e^9$ prediction. Pfenninger's measurement has been added for orientation purposes. Clearly, if $R_a$ is very great, the flow begins as ordinary Blasius flow, because the

† This is a controversial assumption, because the flow has been proved stable only for rotationally symmetric disturbances. Little is known of the stability with respect to other types of disturbance.

boundary-layer thickness is so negligible compared to the pipe radius that the core velocity is constant initially. Basically the transition locus varies in a hyperbolic fashion and approaches a vertical asymptote, which is $R_{a\,\mathrm{crit}}$.

The above facts and considerations indicate the following sequence of events. The flow begins as a very thin boundary layer. At first it is so thin that it is stable

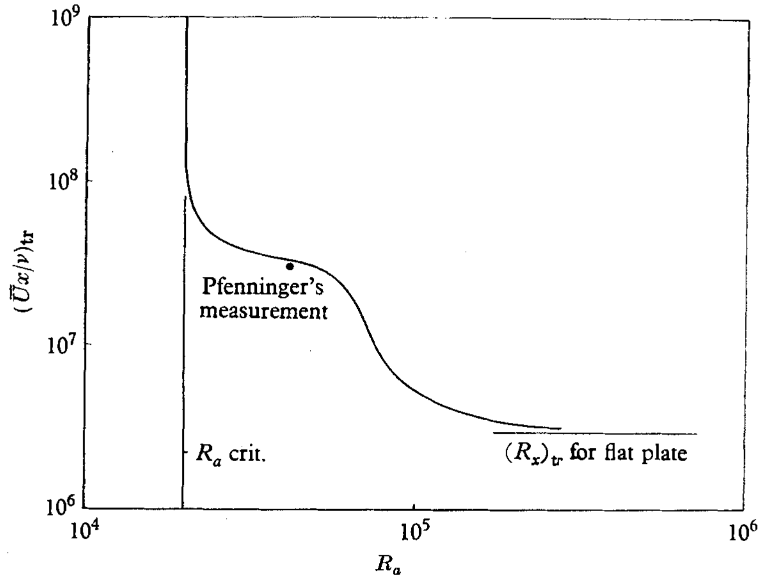


FIGURE 7. Nature of the transition locus in the $R_a$ *vs* $R_x$ plane. ●, Pfenninger's measurement.

to small disturbances. But shortly, as it grows, it becomes unstable and disturbances begin to amplify. If the flow is such that the tube Reynolds number is greater than the critical value, the disturbances grow until they undergo an apparent amplification ratio of about $e^9$, at which point transition occurs. If the tube Reynolds number is reduced, the point of transition moves farther downstream, either in $x$ measure or $x/aR_a$ measure. Finally, a Reynolds number will be reached at which an amplification ratio of magnitude $e^9$ is just attained. Below the value $e^9$, waves will grow for a while, but then they will enter the more stable flow farther downstream and will die out. An experiment in which Reynolds number is varied by reducing the velocity in a particular tube would show the transition point moving downstream slowly at first. But when sufficient disturbance amplification could no longer be reached, the point of demarcation between the laminar and turbulent regions would wash downstream to the very end of the tube. The value of $R_x$ for transition would then have a discontinuity in its variation with $R_a$.

Experiments seem to indicate a discontinuity, but they have not been sufficiently thorough to settle the question. Whether the discontinuity could truly exist cannot be answered by theory because damping rates are known to be a function not only of disturbance amplitude but also of wave length. Disturbances other than two-dimensional or rotationally symmetric ones would further modify the picture, and the process just described is by no means the only one by which turbulent flow can develop.

The old question of whether or not the inlet portions of the flow may be neglected can be made to appear absurd by a little consideration. Consider flow in a tube of 2 in. diameter (the same as Pfenninger's). The inlet length ends at $x/aR_a$ of about 0·26, say 0·24 to make the arithmetic easier. Then for a 2 in. tube, the end of the inlet is at $x = 0{\cdot}24aR_a$ or $x = (0{\cdot}24 \div 12)\,R_a = 0{\cdot}02\,R_a$. If $R_a = 10^4$, $x = 200$ ft. The first 200 ft. of any laminar flow can hardly be ignored!

## 6. The $e^9$ correlation plot

In Pfenninger's test the turbulence was very low and the tube was very smooth. Thus his measurement qualifies for inclusion in the correlation plot of Smith & Gamberoni (1956). This plot compares measured transition points with ones calculated by assuming transition occurs when Tollmien–Schlichting waves are

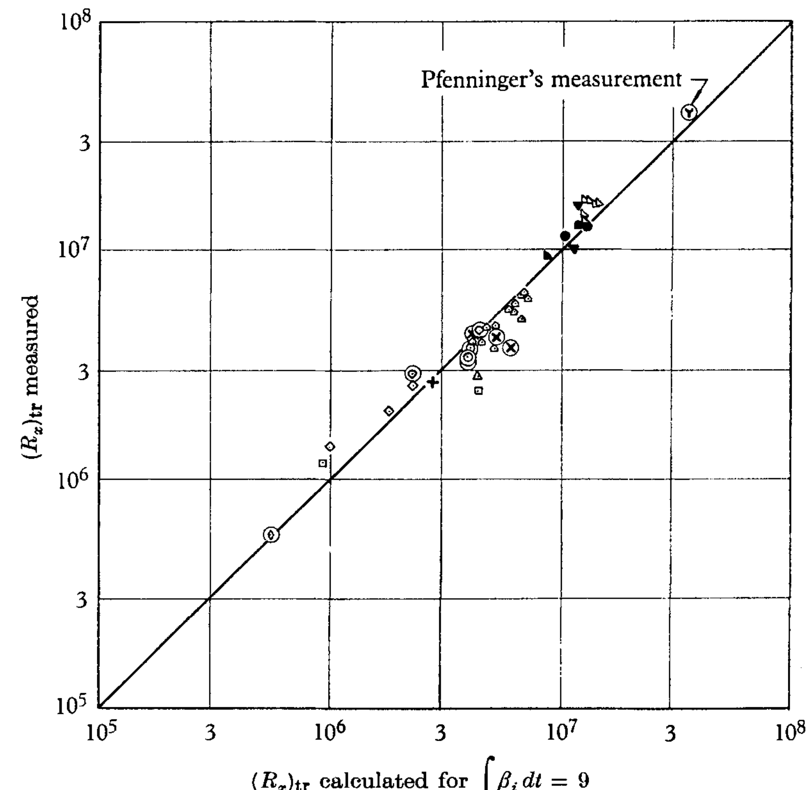


FIGURE 8. Transition data, correlation between measurement and prediction by stability theory. Data from Smith & Gamberoni (1956) except for Pfenninger's measurement.

| Standard deviation: Type of body | No. of points | % deviation |
|---|---|---|
| 2-dimensional | 31 | 11·5 |
| 3-dimensional | 10 | 17·2 |

*Note*: (1) Solid points represent flight-test data. (2) Circled points represent bodies of revolution. (3) Open points represent wind-tunnel data. $(R_x)_{tr} = (U_{tr}x_{tr})/\nu$.

amplified by a factor of $e^9$. The result is shown in figure 8. The numerical values are $(R_x)_{tr} = 40{\cdot}4 \times 10^6$, with $U_{tr} = 35{\cdot}17\,\mathrm{m/sec}$, $x_{tr} = 18\,\mathrm{m}$, $\nu = 15{\cdot}65 \times 10^{-6}$ $\mathrm{m^2/sec}$. The Reynolds number $(R_\theta)_{tr}$, based on momentum thickness, is approximately 3600. The remaining points can be identified by reference to the earlier work. Considering the rough method of calculation, the agreement between measurement and prediction is, to say the least, a remarkable coincidence. Here $R_x$ is based on the local core velocity $U_c$, not $\overline{U}$. The additional point increases the range of validity of the correlation appreciably, from a transition value of $R_x$ of about 17 million to over 40 million. The correlation now covers a range of Reynolds numbers $R_x$ for transition from about 570,000 to slightly over 40 million. Both extremes are axially symmetric flow, the lower being flow about a sphere, the upper being flow inside a round tube. Standard deviations for the sets of points are included. The values are not consistent with those shown in the earlier work, because there was an error in the original calculations.

## 7. Conclusions

1. Existing experimental data for the flow in a round tube are inadequate and unsatisfactory.

2. Existing theoretical solutions for the velocity distribution in the inlet of a round tube are of insufficient accuracy, at least for the requirements of stability theory.

3. Provided the boundary-layer thickness is less than about half the tube radius, conventional boundary-layer calculations can predict velocity profiles along the inlet of a round tube with about the same accuracy as the more elaborate methods.

4. The existence of turbulent flow in a smooth round tube can be explained by consideration of the inlet region. Transition can readily occur in this portion because the velocity profiles have a finite stability limit.

5. Apparently the $e^9$ method will predict the location of transition in the inlet length of a smooth round tube throughout a significant fraction of its length.

6. At transition, the apparent amplification factor for Tollmien–Schlichting waves is found to be a constant over a very large range of Reynolds numbers and variety of flows, from $R_x = 570{,}000$ to over 40 million. The constant factor is about $e^9$, regardless of body shape.

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